

# THE RODRIGUEZ-VILLEGAS TYPE CONGRUENCES FOR TRUNCATED $q$ -HYPERGEOMETRIC FUNCTIONS

VICTOR J. W. GUO, HAO PAN, AND YONG ZHANG

## 1. INTRODUCTION

Define the truncated hypergeometric function

$${}_2F_1 \left[ \begin{matrix} x_1, x_2 \\ y_1 \end{matrix} \middle| z \right]_n = \sum_{k=0}^{n-1} \frac{(x_1)_k (x_2)_k}{(y_1)_k} \cdot \frac{z^k}{k!},$$

where  $(x)_k = x(x+1)\cdots(x+k-1)$  if  $k \geq 1$  and  $(x)_0 = 0$ . Motivated by the Calabi-Yau manifold, Rodriguez-Villegas [4] conjectured some congruences on truncated hypergeometric functions modulo  $p^2$  and  $p^3$ . Nowadays, most of those conjectures have been confirmed. For example, with help of the Gross-Koblitz formula, Mortenson [3] proved that, for any prime  $p \geq 5$ ,

$${}_2F_1 \left[ \begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| 1 \right]_p \equiv \left( \frac{-1}{p} \right) \pmod{p^2}, \quad (1.1)$$

$${}_2F_1 \left[ \begin{matrix} 1/3, 2/3 \\ 1 \end{matrix} \middle| 1 \right]_p \equiv \left( \frac{-3}{p} \right) \pmod{p^2}, \quad (1.2)$$

$${}_2F_1 \left[ \begin{matrix} 1/4, 3/4 \\ 1 \end{matrix} \middle| 1 \right]_p \equiv \left( \frac{-2}{p} \right) \pmod{p^2}, \quad (1.3)$$

$${}_2F_1 \left[ \begin{matrix} 1/6, 5/6 \\ 1 \end{matrix} \middle| 1 \right]_p \equiv \left( \frac{-1}{p} \right) \pmod{p^2}, \quad (1.4)$$

where  $\left( \frac{\cdot}{p} \right)$  is the Legendre symbol modulo  $p$ . Z.-W. Sun [6] gave an elementary proof for (1.1)–(1.4). Subsequently, Z.-H. Sun [5] generalized the above congruences to the following unified form:

$${}_2F_1 \left[ \begin{matrix} \alpha, 1-\alpha \\ 1 \end{matrix} \middle| 1 \right]_p \equiv (-1)^{\langle -\alpha \rangle_p} \pmod{p^2}. \quad (1.5)$$

Here  $\alpha$  is a rational number whose denominator is prime to  $p$ , and  $\langle x \rangle_p$  denotes the integer in  $\{0, 1, \dots, p-1\}$  such that  $x \equiv \langle x \rangle_p \pmod{p}$ . Note that we can also define  $\langle x \rangle_n$  similarly on condition that the denominator of  $x$  is prime to  $n$ .

It is natural to define the truncated  $q$ -hypergeometric function as follows:

$${}_2\phi_1 \left[ \begin{matrix} x_1, x_2 \\ y \end{matrix} \middle| q, z \right]_n = \sum_{k=0}^{n-1} \frac{(x_1; q)_k (x_2; q)_k}{(y; q)_k (q; q)_k} z^k,$$

where

$$(x; q)_k = \begin{cases} (1-x)(1-xq) \cdots (1-xq^{k-1}), & \text{if } k \geq 1, \\ 1, & \text{if } k = 0. \end{cases}$$

Recently, Guo and Zeng [2] have obtained a  $q$ -analogue of (1.1):

$${}_2\phi_1 \left[ \begin{matrix} q, q \\ q^2 \end{matrix} \middle| q^2, 1 \right]_p \equiv \left( \frac{-1}{p} \right) q^{\frac{1-p^2}{4}} \pmod{[p]^2}, \quad p \geq 3, \quad (1.6)$$

where  $[p] = 1 + q + \cdots + q^{p-1}$  and the above congruence is considered over the polynomial ring  $\mathbb{Z}[q]$ . Furthermore, they also conjectured that, for  $p \geq 5$ ,

$${}_2\phi_1 \left[ \begin{matrix} q, q^2 \\ q^3 \end{matrix} \middle| q^3, 1 \right]_p \equiv \left( \frac{-3}{p} \right) q^{\frac{1-p^2}{3}} \pmod{[p]^2}, \quad (1.7)$$

$${}_2\phi_1 \left[ \begin{matrix} q, q^3 \\ q^4 \end{matrix} \middle| q^4, 1 \right]_p \equiv \left( \frac{-2}{p} \right) q^{\frac{3(1-p^2)}{8}} \pmod{[p]^2}, \quad (1.8)$$

$${}_2\phi_1 \left[ \begin{matrix} q, q^5 \\ q^6 \end{matrix} \middle| q^6, 1 \right]_p \equiv \left( \frac{-1}{p} \right) q^{\frac{5(1-p^2)}{12}} \pmod{[p]^2}. \quad (1.9)$$

In this paper, we shall prove the congruences (1.7)–(1.9). More precisely, we shall give a  $q$ -analogue of (1.5) as follows.

**Theorem 1.1.** *Let  $n, d \geq 2$  with  $\gcd(n, d) = 1$  and let  $r$  be an integer. Then*

$${}_2\phi_1 \left[ \begin{matrix} q^r, q^{d-r} \\ q^d \end{matrix} \middle| q^d, 1 \right]_n \equiv (-1)^a q^{(ad+r)(a-\frac{n-1}{2})-d\binom{a+1}{2}} \pmod{\Phi_n(q)^2}, \quad (1.10)$$

where  $\Phi_n(q)$  denotes the  $n$ -th cyclotomic polynomial in  $q$  and  $a = \langle -r/d \rangle_n$ .

For example, letting  $r = 1$ ,  $d = 3$  and letting  $n = p$  be a prime greater than 3, we have

$$a = \langle -1/3 \rangle_p = \begin{cases} (p-1)/3, & \text{if } p \equiv 1 \pmod{3}, \\ (2p-1)/3, & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

and so

$${}_2\phi_1 \left[ \begin{matrix} q, q^2 \\ q^3 \end{matrix} \middle| q^3, 1 \right]_p \equiv \left( \frac{-3}{p} \right) q^{(ad+r)(a-\frac{n-1}{2})-d\binom{a+1}{2}} = \left( \frac{-3}{p} \right) q^{\frac{1-p^2}{3}} \pmod{[p]_q^2},$$

which is the congruence (1.7).

## 2. PROOF OF THEOREM 1.1

Recall that the  $q$ -binomial coefficients  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{n-k+1}; q)_k}{(q; q)_k},$$

and the  $q$ -integer  $[n]_q$  is defined as  $[n]_q = \frac{1-q^n}{1-q}$ . Our proof of Theorem 1.1 only requires the following two forms of the  $q$ -Chu-Vandemonde identity (see, for example, [1, (3.3.10)]):

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{j=0}^k q^{(n-j)(k-j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k-j \end{bmatrix}_q, \quad (2.1)$$

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{j=0}^k q^{j(m-k+j)} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} m \\ k-j \end{bmatrix}_q. \quad (2.2)$$

It is easy to see that

$$\frac{(q^r; q^d)_k}{(q^d; q^d)_k} = (-1)^k q^{rk+d\binom{k}{2}} \begin{bmatrix} -r/d \\ k \end{bmatrix}_{q^d}.$$

So writing  $\alpha = -r/d$ , the congruence (1.10) is equivalent to

$$\sum_{k=0}^{n-1} q^{dk^2} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^d} \begin{bmatrix} -1-\alpha \\ k \end{bmatrix}_{q^d} \equiv (-1)^a q^{(\alpha-a)d(a-\frac{n-1}{2})-d\binom{a+1}{2}} \pmod{\Phi_n(q)^2}. \quad (2.3)$$

Note that  $a = \langle \alpha \rangle_n$ . Let  $s = (\alpha - a)/n$ . Then  $sd$  is an integer. By the  $q$ -Chu-Vandemonde identity (2.1), we have

$$\begin{aligned} \begin{bmatrix} a+sn \\ k \end{bmatrix}_{q^d} &= \sum_{j=0}^k q^{d(sn-j)(k-j)} \begin{bmatrix} sn \\ j \end{bmatrix}_{q^d} \begin{bmatrix} a \\ k-j \end{bmatrix}_{q^d} \\ &\equiv \begin{bmatrix} a \\ k \end{bmatrix}_{q^d} - \sum_{j=1}^k (-1)^j q^{-dj(k-j)-d\binom{j}{2}} \frac{[sn]_{q^d}}{[j]_{q^d}} \begin{bmatrix} a \\ k-j \end{bmatrix}_{q^d} \pmod{\Phi_n(q)^2}, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \frac{[sn]_{q^d}}{[j]_{q^d}} &= \frac{1-q^{sdn}}{1-q^{jd}} \equiv 0 \pmod{\Phi_n(q)}, \\ \begin{bmatrix} sn-1 \\ j-1 \end{bmatrix}_{q^d} &\equiv \begin{bmatrix} -1 \\ j-1 \end{bmatrix}_{q^d} = (-1)^{j-1} q^{-d\binom{j}{2}} \pmod{\Phi_n(q)}, \end{aligned}$$

for  $1 \leq j \leq n$ . Similarly, there holds

$$\begin{aligned} & \begin{bmatrix} -1 - a - sn \\ k \end{bmatrix}_{q^d} \\ & \equiv \begin{bmatrix} -1 - a \\ k \end{bmatrix}_{q^d} - \sum_{j=1}^k (-1)^j q^{-dj(k-j)-d\binom{j}{2}} \frac{[-sn]_{q^d}}{[j]_{q^d}} \begin{bmatrix} -1 - a \\ k - j \end{bmatrix}_{q^d} \pmod{\Phi_n(q)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{k=0}^{n-1} q^{dk^2} \begin{bmatrix} a + sn \\ k \end{bmatrix}_{q^d} \begin{bmatrix} -1 - a - sn \\ k \end{bmatrix}_{q^d} - \sum_{k=0}^{n-1} q^{dk^2} \begin{bmatrix} a \\ k \end{bmatrix}_{q^d} \begin{bmatrix} -1 - a \\ k \end{bmatrix}_{q^d} \\ & \equiv - \sum_{k=1}^{n-1} q^{dk^2} \begin{bmatrix} a \\ k \end{bmatrix}_{q^d} \sum_{j=1}^k (-1)^j q^{-dj(k-j)-d\binom{j}{2}} \frac{[sn]_{q^d}}{[j]_{q^d}} \begin{bmatrix} -1 - a \\ k - j \end{bmatrix}_{q^d} \\ & \quad - \sum_{k=1}^{n-1} q^{dk^2} \begin{bmatrix} -1 - a \\ k \end{bmatrix}_{q^d} \sum_{j=1}^k (-1)^j q^{-dj(k-j)-d\binom{j}{2}} \frac{[sn]_{q^d}}{[j]_{q^d}} \begin{bmatrix} a \\ k - j \end{bmatrix}_{q^d} \pmod{\Phi_n(q)^2}. \end{aligned} \tag{2.4}$$

By the  $q$ -Chu-Vandemonde identity (2.2), we have

$$\begin{aligned} & \sum_{k=1}^{n-1} q^{dk^2} \begin{bmatrix} a \\ k \end{bmatrix}_q \sum_{j=1}^k \frac{(-1)^j q^{-dj(k-j)-d\binom{j}{2}}}{[j]_{q^d}} \begin{bmatrix} -1 - a \\ k - j \end{bmatrix}_{q^d} \\ & = \sum_{j=1}^a \frac{(-1)^j q^{dj^2-d\binom{j}{2}}}{[j]_{q^d}} \sum_{k=j}^a q^{dk(k-j)} \begin{bmatrix} a \\ a - k \end{bmatrix}_{q^d} \begin{bmatrix} -1 - a \\ k - j \end{bmatrix}_{q^d} \\ & = \sum_{j=1}^a \frac{(-1)^j q^{dj^2-d\binom{j}{2}}}{[j]_{q^d}} \begin{bmatrix} -1 \\ a - j \end{bmatrix}_{q^d} \\ & = (-1)^a \sum_{j=1}^a \frac{q^{-\frac{d(a+1)(a-2j)}{2}}}{[j]_{q^d}}. \end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned}
& \sum_{k=1}^{n-1} q^{dk^2} \begin{bmatrix} -1-a \\ k \end{bmatrix}_{q^d} \sum_{j=1}^k \frac{(-1)^j q^{-dj(k-j)-d\binom{j}{2}}}{[j]_{q^d}} \begin{bmatrix} a \\ k-j \end{bmatrix}_{q^d} \\
& \equiv \sum_{k=1}^{n-1} q^{dk^2} \begin{bmatrix} n-1-a \\ k \end{bmatrix}_{q^d} \sum_{j=1}^k \frac{(-1)^j q^{-dj(k-j)-d\binom{j}{2}}}{[j]_{q^d}} \begin{bmatrix} 1-(n-1-a) \\ k-j \end{bmatrix}_{q^d} \\
& \equiv (-1)^{n-1-a} \sum_{j=1}^{n-1-a} \frac{q^{-\frac{d(n-a)(n-1-a-2j)}{2}}}{[j]_{q^d}} \\
& \equiv (-1)^{n-1-a} \sum_{j=a+1}^{n-1} \frac{q^{-\frac{d(n-a)(2j-n-1-a)}{2}}}{[n-j]_{q^d}} \pmod{\Phi_n(q)}. \tag{2.6}
\end{aligned}$$

If  $n$  is even, then

$$q^{\frac{n}{2}} = -1 + \frac{1-q^n}{1-q^{\frac{n}{2}}} \equiv -1 \pmod{\Phi_n(q)},$$

and (since  $d$  is odd in this case)

$$q^{-\frac{d(n-a)(2j-n-1-a)}{2}} \equiv q^{\frac{da(2j-1-a)-dn(2j-1)}{2}} \equiv -q^{\frac{da(2j-1-a)}{2}} \pmod{\Phi_n(q)};$$

while if  $n$  is odd, then

$$q^{-\frac{d(n-a)(2j-n-1-a)}{2}} = q^{\frac{da(2j-1-a)-dn(2j-1-n)}{2}} \equiv q^{\frac{da(2j-1-a)}{2}} \pmod{\Phi_n(q)}.$$

Hence, we always have

$$(-1)^{n-1-a} \sum_{j=a+1}^{n-1} \frac{q^{-\frac{d(n-a)(2j-n-1-a)}{2}}}{[n-j]_{q^d}} \equiv (-1)^{a-1} \sum_{j=a+1}^{n-1} \frac{q^{-\frac{d(a+1)(a-2j)}{2}}}{[j]_{q^d}} \pmod{\Phi_n(q)}. \tag{2.7}$$

Noticing that

$$\begin{aligned}
\sum_{j=1}^{n-1} \frac{1}{[j]_{q^d}} &= \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{1}{[j]_{q^d}} + \frac{1}{[n-j]_{q^d}} \right) \\
&= \frac{1}{2} \sum_{j=1}^{n-1} \left( \frac{1}{[j]_{q^d}} + \frac{q^{jd}}{[n]_{q^d} - [j]_{q^d}} \right) \\
&\equiv \frac{1}{2} \sum_{j=1}^{n-1} \frac{1-q^{jd}}{[j]_{q^d}} = \frac{n-1}{2} (1-q^d) \pmod{\Phi_n(q)},
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{j=1}^{n-1} \frac{q^{d(a+1)j}}{[j]_{q^d}} &= \sum_{j=1}^{n-1} \frac{1}{[j]_{q^d}} - (1 - q^d) \sum_{j=1}^{n-1} \sum_{k=0}^a q^{dkj} \\
&= \sum_{j=1}^{n-1} \frac{1}{[j]_{q^d}} - (n-1)(1 - q^d) - (1 - q^d) \sum_{k=1}^a \frac{q^{dk} - q^{dkn}}{1 - q^{dk}} \\
&\equiv \frac{n-1}{2}(1 - q^d) - (n-1)(1 - q^d) + a(1 - q^d) \pmod{\Phi_n(q)}. \quad (2.8)
\end{aligned}$$

It follows from (2.4)–(2.8) that

$$\begin{aligned}
&\sum_{k=0}^{n-1} q^{dk^2} \begin{bmatrix} a + sn \\ k \end{bmatrix}_{q^d} \begin{bmatrix} -1 - a - sn \\ k \end{bmatrix}_{q^d} \\
&\equiv \sum_{k=0}^{n-1} q^{dk^2} \begin{bmatrix} a \\ k \end{bmatrix}_{q^d} \begin{bmatrix} -1 - a \\ k \end{bmatrix}_{q^d} + \frac{2a+1-n}{2} (-1)^a q^{-d\binom{a+1}{2}} (1 - q^{sdn}) \\
&= \begin{bmatrix} -1 \\ a \end{bmatrix}_{q^d} + \frac{2a+1-n}{2} (-1)^a q^{-d\binom{a+1}{2}} (1 - q^{sdn}) \pmod{\Phi_n(q)^2}
\end{aligned}$$

by the  $q$ -Chu-Vandemonde identity (2.2). Noticing that  $\begin{bmatrix} -1 \\ a \end{bmatrix}_{q^d} = (-1)^a q^{-d\binom{a+1}{2}}$  and

$$\begin{aligned}
q^{(\alpha-a)d\left(\frac{n-1}{2}-a\right)} &= (1 - (1 - q^{sdn}))^{\frac{n-1}{2}-a} \\
&\equiv 1 + \frac{2a+1-n}{2} (1 - q^{sdn}) \pmod{\Phi_n(q)^2},
\end{aligned}$$

we complete the proof.  $\square$

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DEPARTMENT OF MATHEMATICS, SHANGHAI KEY LABORATORY OF PMMP, EAST CHINA  
NORMAL UNIVERSITY, 500 DONGCHUAN RD., SHANGHAI 200241, PEOPLE'S REPUBLIC OF  
CHINA

*E-mail address:* jwguo1977@aliyun.com

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S RE-  
PUBLIC OF CHINA

*E-mail address:* haopan79@zoho.com

DEPARTMENT OF BASIC COURSE, NANJING INSTITUTE OF TECHNOLOGY, NANJING 211167,  
PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* yongzhang1982@163.com